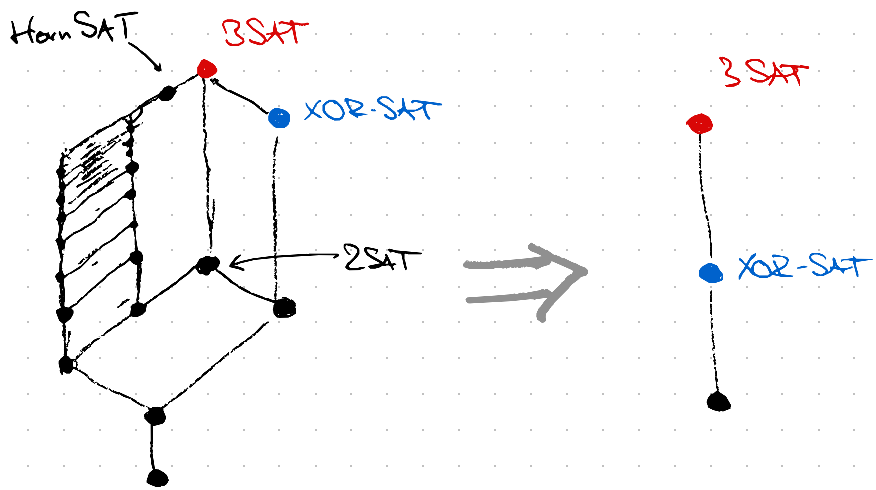
Local consistency as a reduction between Constraint Satisfaction Problems

Victor Dalmau & Jakub Opršal



LICS 2024

Constraint satisfaction problem

Can you satisfy a bunch of constraints simultaneously?

$$x + 2y \leq 2$$
$$y + z + w \leq 1$$
$$-3x - 4w \leq -3$$

CSP

Satisfaction of primitive positive formula

fix $\mathbf{B} = (B; R, S, \ldots)$

$\exists x_1, \ldots, x_l$ s.t.

$R(x_2, x_3) \wedge S(x_1, x_2, x_4)$

$\wedge\ R(x_4, x_2) \wedge x_2 = x_3$

3SAT

lin3SAT

3-colouring

$\mathbb{Z}_2$-affine systems (XOR-SAT)

2-colouring

Linear programming

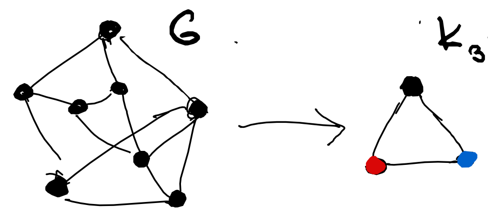
Horn-3SAT

Conjunctive queries in databases

Homomorphism problem

Fix $\mathbf{B}$, input: $\mathbf{A}$

$\mathbf{A} \xrightarrow{?} \mathbf{B}$



Algebraic approach & gadget reductions.

theorem [Bulatov, Jeavons, Krokhin, '05; Barto, O., Pinsker, '17]
$\underline{A}, \underline{B}$ - finite rel. structures. TFAE:

1. $CSP(\underline{A}) \leq_L CSP(\underline{B})$ via a simple gadget reduction
2. $Pol(\underline{B}) \rightarrow Pol(\underline{A})$

theorem [Feder-Vardi conjecture, '98; Bulatov '17 & Zhuk '17]
$\underline{A}$ - finite rel. structure

1. either $Pol(\underline{A}) \rightarrow \mathcal{P}$ & $CSP(\underline{A})$ is NP-complete,
2. or $Pol(\underline{A}) \not\rightarrow \mathcal{P}$ & $CSP(\underline{A})$ is in P.

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2. $Pol(\underline{B}) \to Pol(\underline{A})$

ALL NP-HARDNESS IS EXPLAINED BY GADGET REDUCTIONS!

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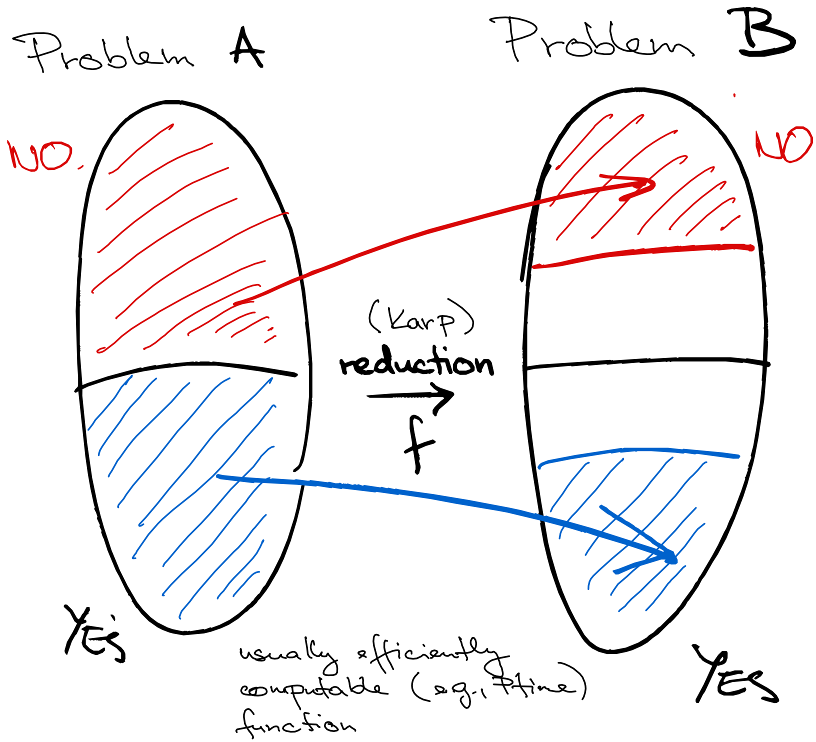
Which <u>promise</u> CSPs are NP-complete?

What is the class of reductions that is sufficient to provide all known NP-hardness in promise CSPs?

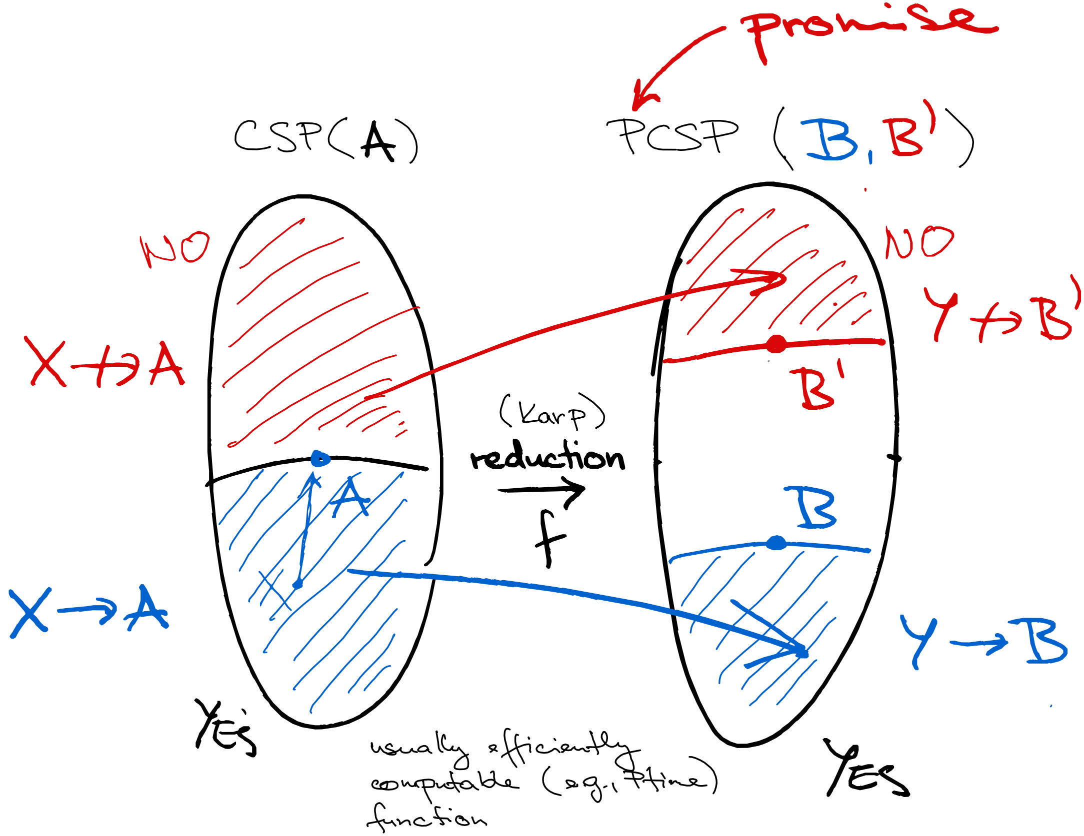
Reductions & Promises



DEFINITION

A **promise** problem is to decide between <u>disjoint</u> (but not necessarily complementary) cases Yes / No

Reductions & Promises



– Approximate graph colouring.
given a graph G that is k-colourable
find a colouring with c colours!
(where $c > k$) $\mathrm{PCSP}(K_k, K_c)$

– $\mathrm{PCSP}(K_3, K_4)$ is NP-hard
[Khanna, Linial, Safra, 2000]
[Guruswami, Khanna, 2004]
[Brakensiek, Guruswami, 2016]

– $\mathrm{PCSP}(K_3, K_5)$ is NP-hard
[Bulín, Krokhin, O, 2019]

– $\mathrm{PCSP}(K_{2d}, K_{\binom{2d}{d}-1})$ is NP-hard
[Wrochna, Živný, 2020]

– $\mathrm{PCSP}(K_3, K_6)$ is open!

– Promise 1-in-3SAT
$\text{1in3} = (\{0,1\}, \{(001),(010),(100)\})$
Fix B s.t. $\text{1-in-3} \to B$, given
a solvable 1in3SAT instance, find
a homomorphism to B.

$\mathrm{PCSP}(\text{1in3}, B)$

[Barto, Battistelli, Berg, 2021]

– $\mathrm{PCSP}(\text{1in3}, \text{NAE}) \in \mathrm{P}$
[Brakensiek, Guruswami, 2018]

Conjecture [Nakajima, Živný, 2023]
$\mathrm{PCSP}(\text{1in3}, B)$ is NP-hard
unless $\mathbb{Z} \to B$
where $\mathbb{Z} = (\mathbb{Z};\ x+y+z=1)$

[Ciardo, Kozik, Krokhin, Nakajima, Živný, LICS 2024]
1in3 vs NAE: Dichotomy of a broken promise ← TODAY @ 12PM!

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GADGET REDUCTIONS DO NOT SUFFICE!

- Promise 1-in-3SAT
  1in3 = $(\{0,1\}, \{(001), (010), (100)\})$
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[Barto, Kozik, '22]

Theorem.

If there is a weak homomorphisms

$\mathrm{Pol}(\underline{A}, \underline{A}') \xrightarrow{dir} \mathrm{Pol}(\underline{B}, \underline{B}')$

then

$\mathrm{CSP}(\underline{B}, \underline{B}') \leq_L \mathrm{CSP}(\underline{A}, \underline{A}')$

[Wrochna, Živný, '20]

Theorem

Coloring graphs that are promised to be k-colourable with $\binom{k}{\lfloor k/2 \rfloor} - 1$ colours is NP-hard.

(via arc-digraph)

k-consistency reduction $\equiv$ Datalog reductions

(Def 3.10) (Def 3.6)

Theorem 2. A PCSP($\underline{A}$, $*$) reduces to PCSP($\underline{B}$, $*'$) via k-consistency reduction iff via Datalog reduction.

FOR ALL KNOWN NP-COMPLETE PROMISE CSPS Γ,
IT IS POSSIBLE THAT 3-COLOURING $\leq_{DL}$ PCSP(Γ)!

GADGET REDUCTIONS

$\Gamma \leq_{\text{gadget}} \Delta$

DATALOG = k-CONSISTENCY REDUCTIONS

$\Gamma \leq_{DL} \Delta$

$\mathrm{Pol}(\Gamma) \longrightarrow \mathrm{Pol}(\Delta)$

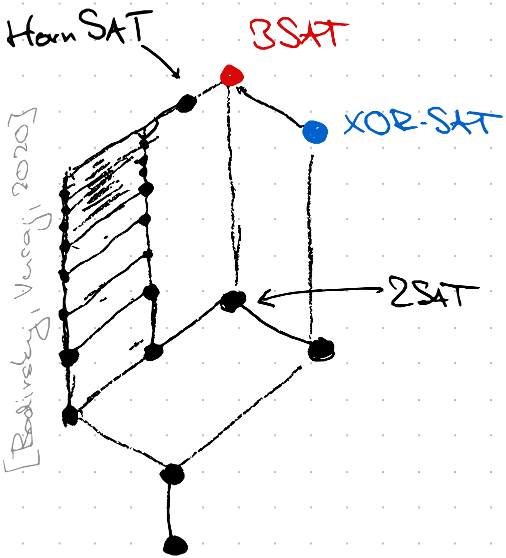
$?(\Gamma) \xrightarrow{?} ?(\Delta)$

To better understand the tractability side of the CSP dichotomy

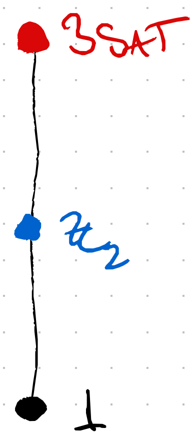
Reduce all tractable to a few special cases.

Boolean CSPs / Reductions

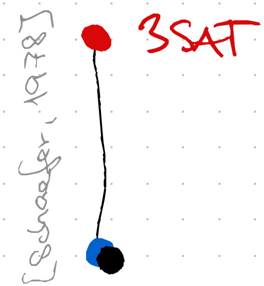
gadget



Datalog$^{\cup}$



P-time
(P ≠ NP)



CONJECTURE 1.

For every finite template $\mathbb{A}$ either:

① 3-colouring $\leq_{DL}$ CSP$(\mathbb{A})$

or:

② CSP$(\mathbb{A}) \leq_{DL}$ CSP$(\mathbb{Z})$

where CSP$(\mathbb{Z})$ denotes solving systems of linear equations over $\mathbb{Z}$.

A COUNTEREXAMPLE EARNS YOU WHISKY!

+ Cohomological k-consistency [Ó Conghaile 2023]
+ The hierarchy of BLP+AIP [Brakensiek, Guruswami, Wrochna, Živný 2020]
+ Higher levels of CLAP [Ciardo, Živný 2022]

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Thank you!

[Atserias, Bulatov, Dawar, '05]

theorem

If $\mathrm{Pol}(\underline{B}) \to \mathrm{Pol}(\underline{A})$
then $\mathrm{CSP}(\underline{A}) \leq \mathrm{CSP}(\underline{B})$
via a Datalog interpretation
(with parameters).

Our Proof

[Kolaitis, Vardi, '00]

theorem

A CSP is solved by
the local consistency algorithm
iff its complement is expressible
in Datalog

+ the algebraic approach
to promise CSP
[Barto, Bulín, Krokhin, O., 2021]