

ALGEBRAIC APPROACH TO CONSTRAINT SATISFACTION PROBLEMS

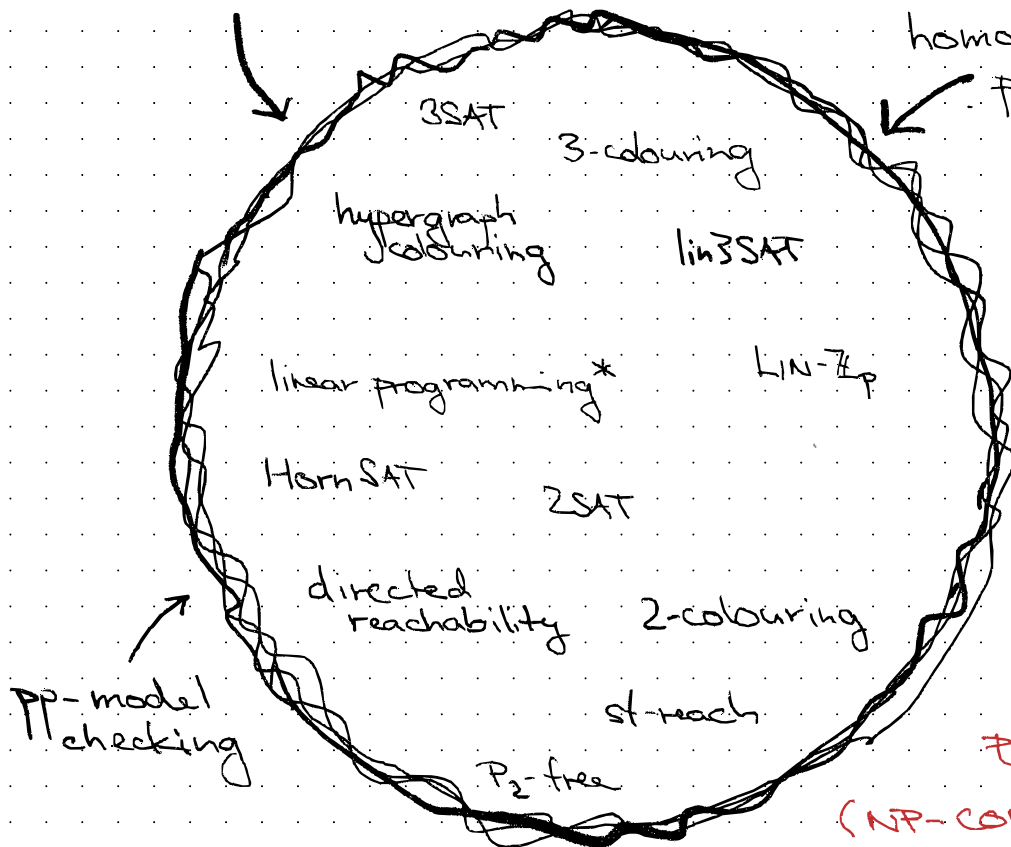
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satisfying local
constraints

CONSTRAINT
SATISFACTION
PROBLEMS

homomorphism
problem



WHY ARE SOME
PROBLEMS HARD
(NP-COMPLETE), WHILE
OTHER ARE EASY (IN P)?

CONSTRAINT SATISFACTION

Given a list of **constraints** over some **variables**, assign values from a some **domain**, so that all constraints are satisfied.

Example 1. 3SAT

$$\underbrace{(x \vee \neg y \vee z)}_{\text{constraint}} \wedge (y \vee z) \wedge (x \vee \neg w \vee \neg z)$$

↖ ↗
variables

domain: $\{ \text{TRUE}, \text{FALSE} \}$

Example 2. LIN($\mathbb{Z}_3$)

domain: $\{0, 1, 2\}$

$$\begin{aligned} \boxed{x + 2y} &= 1 && \text{constraint} \\ 2y + w &= 2 \\ x - z + w &= 0 && \text{variable} \end{aligned}$$

Example 3. HORN3SAT

$$(x \wedge y \rightarrow z) \wedge (y \rightarrow \text{FALSE}) \wedge x$$

Example 4. Linear programming

domain: $\mathbb{Q}$

$$\begin{aligned} x + 2y &\leq 1 \\ 2y + w &= 2 \\ x - y + z &\geq 0 \end{aligned}$$

CONSTRAINT SATISFACTION

ex. 3SAT

$$(x \vee y \vee z) \wedge (y \vee w) \wedge (\neg z \vee x)$$

VERIFICATION OF PRIMITIVE POSITIVE FORMULAE

Fix a structure $\underline{A}$, given
pp-sentence ϕ , decide
whether $\underline{A} \models \phi$.

ex. $\underline{A} = (\{0, 1\}, 0, 1)$

$$\phi = \exists x, y, z (x=0) \wedge (x=y) \\ \wedge (y=z) \wedge (z=1)$$

HOMOMORPHISM PROBLEMS

Fix a relational structure $\underline{A}$
(e.g. a graph).

Given $\underline{B}$, decide if there
is a homomorphism

$$\underline{B} \rightarrow \underline{A}$$

ex. graph 3-colouring

$$\underline{A} = K_3$$

These three
perspectives
are equivalent*.

* up to logspace
reductions

For the rest of this lecture,

A is a finite relational structure.

Fix a relational language:
 $R, S, \dots$

A template is a structure

$$\underline{A} = (\underline{A}, R^{\underline{A}}, S^{\underline{A}}, \dots)$$

where:

- $\underline{A}$ is a set
- $R^{\underline{A}} \subseteq \underline{A}^{\text{arity}(R)}$, etc.

CSP(A):

Input: another structure B

Goal: decide if

$$\underline{B} \rightarrow \underline{A}$$

I WILL NOT
TALK ABOUT:

- Optimisation
(= Valued CSP)
e.g. MaxCut

- Approximation
(= Promise VCSP)
e.g. the $\mathbb{RP}$ theorem

- Infinite template
(& Bodirsky-Pinsker)
e.g. directed acyclicity

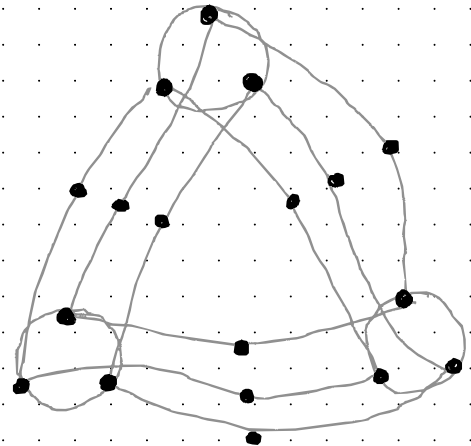
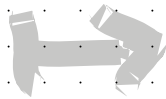
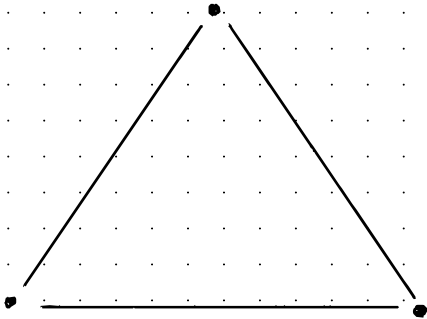
EXAMPLE: A REDUCTION FROM 3-COLORING TO lin3SAT .

3-colouring $\equiv \text{CSP}(K_3)$

lin3SAT :

template: $(\{0,1\}, \mathbb{R}^{\text{lin3}})$

$\mathbb{R}^{\text{lin3}} = \{001, 010, 100\}$



Such reductions were used by Hell and Nešetřil (1990) to provide complexity classification of graph homomorphism problems.

THE COMPLEXITY OF A CSP ONLY
DEPENDS ON THE POLYMORPHISMS
OF ITS TEMPLATE!

— Everyone, everywhere, all at once.

POLYMORPHISMS

def. An n -ary polymorphism of $\underline{A}$ is a homomorphism

$$f: \underline{A}^n \rightarrow \underline{A}$$

example. If $\underline{A}$ is a graph, we want
 $f: V(\underline{A})^n \rightarrow V(\underline{A})$
such that for every n -tuple
of edges $u_1v_1, \dots, u_nv_n$
 $f(u_1, \dots, u_n) f(v_1, \dots, v_n)$
is an edge.

$$\begin{array}{l} f(u_1, u_2, \dots, u_n) = u \\ f(v_1, v_2, \dots, v_n) = v \end{array}$$

Exercises:

1. polymorphisms of K_3
2. polymorphisms of $\text{Lin}(\mathbb{Z}_p)$
3. polymorphisms of Horn SAT

The symbol $\text{pol}(\underline{A})$ denotes the ~~relation~~
of polymorphisms of $\underline{A}$.

BULATOV-ZHUK THEOREM

Theorem. let A be a finite relational structure, then either:

- A satisfies (P) , and $CSP(A)$ is in P , or
- $CSP(A)$ is NP-complete.

The following are equivalent formulations of (P) .

1. A has a polymorphism satisfying

$$s(xyxyz) = s(yxzyz)$$

SIGGERS

2. A has a polymorphism satisfying

$$w(yx \dots x) = w(xy \dots x) = w(x \dots xy)$$

WNU

3. A has a polymorphism satisfying

$$c(x_1 \dots x_p) = c(x_2 \dots x_p x_1)$$

CYCLIC

4. A has a polymorphism satisfying

$$t(x * \dots *) = t(y * \dots *)$$

$$t(* x * *) = t(* y * *)$$

$$t(* \dots * x) = t(* \dots * y)$$

TAYLOR

THEOREM

If A has **less** polymorphisms than B ,
then $CSP(B)$ is **easier** than $CSP(A)$.

More precisely, if $pol(A) \rightarrow pol(B)$, then $CSP(B) \leq_L CSP(A)$.

1977

TAYLOR

Varieties obeying homotopy laws

SCHAEFER

1978

Dichotomy of Boolean
CSPs.

HEU & NEŠETŘIL

1990

Dichotomy of graph
CSPs.

1997

JEAVONS

$\text{pol}(A) : \text{set}$

FEDER & VARDI

1998

Dichotomy conjecture

BULATOV, JEAVONS, KEOKHIN

2001

$\text{pol}(A) : \text{clone}$

2001

BULATOV

Algorithm for Malcev.

BULATOV, JEAVONS, KROKHIN
 $\text{pol}(*)$: clone

2005

BULATOV & DALMAU

Better alg. for Malcev

LAROSE & ZADORI

Bounded width conjecture.

REINGOLD

Undirected reachability is in L.

2008

HAROTI & MCKENZIE

WNUS

BARTO & KOZIK

2014

Bounded width theorem.

BARTO

$\text{pol}(*)$: $\text{Fin} \rightarrow \text{Fin}$

2017

BULATOV; ZHUK

2017

AUSTRIAN, HÅSTAD,
GURUSWAMI
Promise constraint
satisfaction

BARTO, O, PINSKER
The wonderland of reflections
 $\text{pol}(\star) : \text{Fin} \rightarrow \text{Fin}$

(BARTO), BULIN, KROKHIN, O
Algebraic approach to
 promise constraint sat.

2026

BARTO, HADEK, ZHUK
Towards a uniform alg.
a.k.a. "the revolution"

HADEK, JAKL, O
Categorical perspective
on constraint satisfaction

THEOREM.

If there is a natural transformation

$$\text{pol}(\underline{A}) \rightarrow \text{pol}(\underline{B})$$

then

$$\text{CSP}(\underline{B}) \leq_L \text{CSP}(\underline{A}).$$

COROLLARY.

If there is a natural transformation

$$\text{pol}(\underline{A}) \rightarrow \text{Id}$$

then

$\text{CSP}(\underline{A})$ is NP-complete.

$$\begin{aligned} \text{pol}(\underline{A}) : \text{Fin} &\rightarrow \text{Fin} \\ N &\mapsto \text{hom}(\underline{A}^N, \underline{A}) \end{aligned}$$

on maps:

$$N \xrightarrow{\pi} M$$

$$\begin{aligned} \text{hom}(\underline{A}^N, \underline{A}) &\rightarrow \text{hom}(\underline{A}^M, \underline{A}) \\ f &\mapsto \lambda a. f(a \circ \pi) \end{aligned}$$

CATEGORICAL

PROMISE

~~ALGEBRAIC~~ APPROACH TO CONSTRAINT SATISFACTION PROBLEMS

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PROMISES

There is a dense homomorphism

$$\text{pol}(K_3) \rightarrow \text{pol}(3\text{SAT})$$

hence by [Bulatov, Jeavons, Krokhin, 2005]

a logspace reduction:

$$3\text{SAT} \xrightarrow{G} 3\text{-colouring}$$

Observation. [Brakensie, Guruswami, 2016]

THE REDUCTION PROVIDED BY BULATOV ET AL.

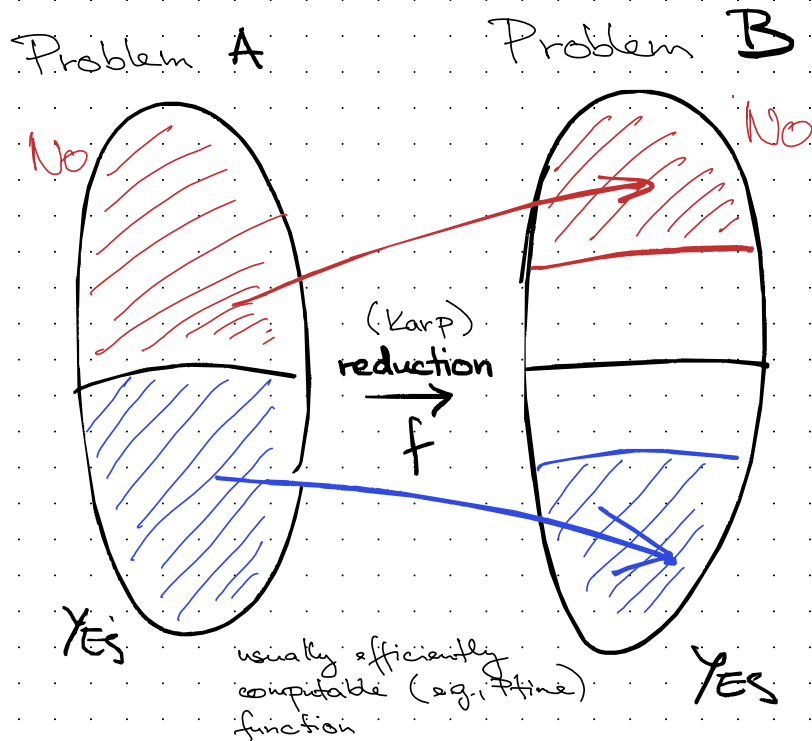
NEVER PRODUCES A GRAPH WITH

CHROMATIC NUMBER 4, i.e.

either $G(\phi) \rightarrow K_3$ (if ϕ is satisfiable)

or $G(\phi) \not\rightarrow K_4$ (if not)

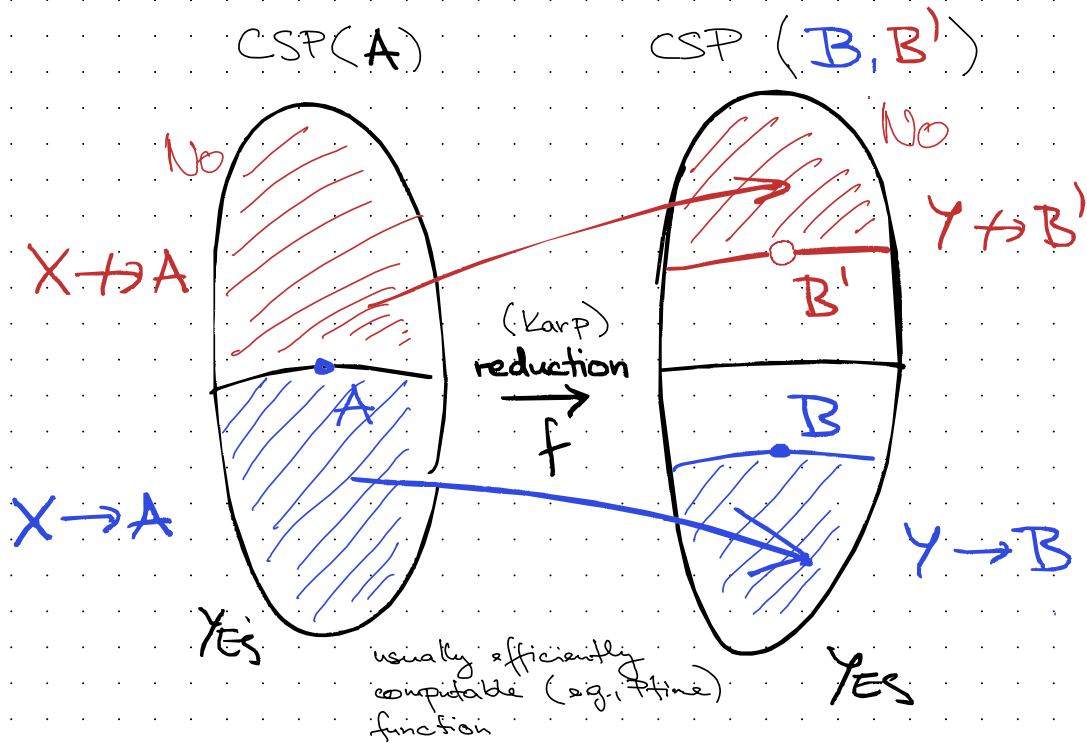
REDUCTIONS AND PROMISES



DEFINITION

A **promise** problem is to decide between disjoint (but not necessarily complementary) cases
Yes / No

REDUCTIONS AND PROMISES



PROOF.

THEOREM.

If there is a natural transformation

$$\text{pol}(\underline{A}) \longrightarrow \text{pol}(\underline{B})$$

then

$$\text{CSP}(\underline{B}) \leq_L \text{CSP}(\underline{A}).$$

PROOF SKETCH.

$$\text{CSP}(\underline{B}) \leq \text{CSP}(\text{Id}, \text{pol}(\underline{B})) \leq \text{CSP}(\text{Id}, \text{pol}(\underline{A})) \leq \text{CSP}(\underline{A})$$

LEMMA

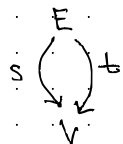
$$F_M(\underline{A}) \rightarrow \underline{B} \quad \simeq \quad \Gamma \rightarrow \text{pol}(\underline{A}, \underline{B})$$

free structure $\uparrow$

$\uparrow$
2nd. $\text{hom}(\underline{A}^N, \underline{B})$

FREE STRUCTURE

Example. A graph $G = (V(G), E(G))$ can be viewed
as a functor $E \rightarrow \mathbf{Fin}$
where E is:



Given $G: E \rightarrow \mathbf{Fin}$ and $M: \mathbf{Fin} \rightarrow \mathbf{Fin}$
the **free graph** $F_H(G)$ is $M \circ G: E \rightarrow \mathbf{Fin}$

LEMMA. For any $A, B: E \rightarrow \mathbf{Fin}$, where
 E is a finite category,
 $\text{pol}(A, B) = \text{Ran}_A B$

PROOF.

THEOREM.

If there is a natural transformation

$$\text{pol}(\underline{A}) \longrightarrow \text{pol}(\underline{B})$$

then

$$\text{CSP}(\underline{B}) \leq_L \text{CSP}(\underline{A}).$$

PROOF SKETCH.

$$\text{CSP}(\underline{B}) \leq \text{CSP}(\text{Id}, \text{pol}(\underline{B})) \leq \text{CSP}(\text{Id}, \text{pol}(\underline{A})) \leq \text{CSP}(\underline{A})$$

PROOF.

THEOREM.

If there is a natural transformation

$$\text{pol}(\underline{A}) \longrightarrow \text{pol}(\underline{B})$$

then

$$\text{CSP}(\underline{B}) \leq_L \text{CSP}(\underline{A}).$$

PROOF SKETCH.

$$\text{CSP}(\underline{B}) \leq \text{CSP}(\text{Id}, \text{pol}(\underline{B})) \leq \text{CSP}(\text{Id}, \text{pol}(\underline{A})) \leq \text{CSP}(\underline{A})$$

□

THE EVOLUTION

Instead of $\text{CSP}(A)$ study $\text{CSP}(\text{Id}, M)$
for a suitable $M: \text{Fin} \rightarrow \text{Fin}$.

(You may assume that M is a monad
if you are not interested in promises.)

OPEN PROBLEM

Let $M: \text{Fin} \rightarrow \text{Fin}$
be a monad, s.t.
there is no natural tran.

$$M \rightarrow \text{Id}$$

then

$\text{CSP}^*(\text{Id}, M)$
is solvable in P time.

$$\text{CSP}^*(\text{Id}, M):$$

Input:

$$D: \mathcal{E} \rightarrow \text{Fin}$$

Decide whether

$$\lim M \circ D \neq \emptyset$$

THEOREM.

If there is a natural transformation

$$\text{pol}(A) \rightarrow \text{pol}(B)$$

then

$$\text{CSP}(B) \leq_L \text{CSP}(A).$$

LEMMA.

$$\text{pol}(A) = \text{Ran}_A A.$$

FURTHER READING:

Hadel, Jael, O. (2016)

A categorical perspective on constraint satisfaction

Barto, Krackin, Willard (2017)

Polymorphisms, and how to use them.

<https://jakub-opreal.info/talks/esslli/>

