Algebraic view on promise constraint satisfaction and hardness of coloring a D-colorable graph with $2D-1$ colors

Jakub Opršal (TU Dresden)

Joint work with Jakub Bulín and Andrei Krokhin

Dagstuhl, June 5, 2018

European Research Council
Established by the European Commission

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 681988)

CSP(**A**)

1. $\mathrm{Pol}(\mathbf{A})$ [Jeavons, Cohen, Gyssens, “97]
2. identities in $\mathrm{Pol}(\mathbf{A})$ [Bulatov, Jeavons, ’01; BJK05]
3. height 1 identities in $\mathrm{Pol}(\mathbf{A})$ [Barto, Pinsker, O, ‘17]

Identity is of height 1 if it is of the form:

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \approx g(x_{\pi(1)}, \ldots, x_{\pi(m)}).$$

$(\sigma\colon [n] \to [k], \pi\colon [m] \to [k])$

No composition!

PCSP($\mathbf{A}$, $\mathbf{B}$):

1. $\mathrm{Pol}(\mathbf{A}, \mathbf{B})$ [Austrin, Håstad, Guruswami, '14; BG16a]
2. **??**

Excuses

~~Polymorphisms of a pair of structures cannot be composed!~~

~~We don't have clones, therefore there are no algebras involved!~~

3. height 1 identities in $\mathrm{Pol}(\mathbf{A}, \mathbf{B})$

$\mathrm{Pol}(K_d, K_{2d-2})$ is equationally trivial [Brakensiek, Guruswami, '16b].

Identities and the main theorem

A Mal'cev condition is a finite set of identities (functional equations).

Example.

$$o(x, x, y, y, y, x) \approx s(x, y)$$
$$o(x, y, x, y, x, y) \approx s(x, y)$$
$$o(y, x, x, x, y, y) \approx s(x, y)$$

Function symbols are variables! I.e., we usually ask for functions that satisfy the identities.

Theorem

If every height 1 Mal'cev condition satisfied by $\mathrm{Pol}(\mathbf{A}, \mathbf{B})$ *is satisfied in* $\mathrm{Pol}(\mathbf{C}, \mathbf{D})$ *then* $\mathrm{PCSP}(\mathbf{C}, \mathbf{D})$ *is log-space reducible to* $\mathrm{PCSP}(\mathbf{A}, \mathbf{B})$.

Example: Graph coloring from hypergraph coloring

Claim

It is NP-hard to distinguish between a graph that is 3-colorable and one that is not 5-colorable. Equivalently, $\mathrm{PCSP}(K_3, K_5)$ is NP-hard.

Theorem (Dinur, Regev, Smyth, '05)

For each $K \geq 2$, it is NP-hard to distinguish between a 3-uniform hypergraph that is colorable by 2 colors, and one that is not colorable by K colors. Consequently, $\mathrm{PCSP}(\mathrm{NAE}_2, \mathrm{NAE}_K)$ is NP-hard for all K.

NAE_k is a relational structure with universe $[k]$ and a single ternary relation R_k saying 'the three entries are not all equal', i.e.,

$$R_k = \{(x, y, z) \in [k]^3 : x \neq y \text{ or } x \neq z\}.$$

Key point. Every height 1 Mal'cev condition satisfied in $\mathrm{Pol}(K_3, K_5)$ is satisfied in $\mathrm{Pol}(\mathrm{NAE}_2, \mathrm{NAE}_K)$.

Intermediate problem: Deciding identities

Fix $N > 0$. Let $\mathcal{U}$ and $\mathcal{V}$ be two disjoint sets of function symbols with arities $\leq N$.

MC(N):

Given $(\Sigma, \mathcal{U}, \mathcal{V})$, where Σ is a bipartite minor condition over $\mathcal{U}$ and $\mathcal{V}$ that involves at most N-ary function symbols, decide whether the condition is satisfied by projections.

A bipartite minor Mal'cev condition over $\mathcal{U}$ and $\mathcal{V}$ is a finite set of identities of the form

$$g(x_{\pi(1)}, \ldots, x_{\pi(m)}) \approx f(x_1, \ldots, x_n)$$

for some $\pi\colon [m] \to [n]$, $f \in \mathcal{U}$, and $g \in \mathcal{V}$.

Identities and label cover

Triviality of minor conditions

$(\Sigma, \mathcal{U}, \mathcal{V})$

$$w(x, x, y) \approx s(x, y)$$

Functions $\equiv$ long codes of labels

Long code of $i \in [n]$ is

$$p_i \colon \mathbf{x} \to \mathbf{x}(i)$$

(a.k.a. the i-th projection).

Label cover

(U, V, E, Π)

$$s \xleftarrow{\pi} w \qquad \pi \colon \begin{array}{l} 0 \searrow \\ 1 \longrightarrow x \\ 2 \longrightarrow y \end{array}$$

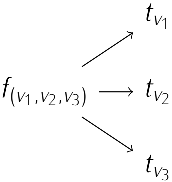
Labels

Commonly used with long code.

Example: From $\mathrm{PCSP}(\mathrm{NAE}_2, \mathrm{NAE}_K)$ to $\mathrm{MC}(6)$

- For each vertex v introduce a binary symbol t_v into $\mathcal{V}$.
- For each edge $e = (v_1, v_2, v_3)$, introduce a 6-ary f_e into U, and add constraints:

$$f_e(x, x, y, y, y, x) \approx t_{v_1}(x, y)$$
$$f_e(x, y, x, y, x, y) \approx t_{v_2}(x, y)$$
$$f_e(y, x, x, x, y, y) \approx t_{v_3}(x, y)$$



Few observations.

- A solution to the MC instance gives a solution to $\mathrm{CSP}(\mathrm{NAE}_2)$.
- It is enough to have a solution in $\mathrm{Pol}(\mathrm{NAE}_2, \mathrm{NAE}_K)$: The assignment $v \mapsto t_v(0, 1)$ is a solution.

Promise satisfaction of identities

Fix N and a set of functions $\mathscr{A}$.

Promise $\mathrm{MC}_{\mathscr{A}}(N)$

Given $(\Sigma, \mathcal{U}, \mathcal{V})$, where Σ is a bipartite minor condition over $\mathcal{U}$ and $\mathcal{V}$ that involves at most N-ary function symbols, decide between:

- Σ is trivial, and
- Σ is not satisfied in $\mathscr{A}$.

Theorem

Let $\mathscr{H}_K = \mathrm{Pol}(\mathrm{NAE}_2, \mathrm{NAE}_K)$. $\mathrm{PMC}_{\mathscr{H}_K}(6)$ *is NP-hard for all* $K \geq 2$.

Theorem

For every PCSP template $(\mathbf{A}, \mathbf{B})$ *there exists* N *such that* $\mathrm{PCSP}(\mathbf{A}, \mathbf{B})$ *is log-space reducible to* $\mathrm{PMC}_{\mathscr{A}}(N)$ *where* $\mathscr{A} = \mathrm{Pol}(\mathbf{A}, \mathbf{B})$.

Example: From PMC to PCSP

Hint

We can ask Is this minor condition satisfied by polymorphisms of a CSP template $\mathbf{A}$? as an instance of $\mathrm{CSP}(\mathbf{A})$.

- For a PCSP template $(\mathbf{A}, \mathbf{B})$, we use just $\mathbf{A}$ to construct the instance.
- Warning! The graph is of exponential size in N.

Theorem

For every PCSP template $(\mathbf{A}, \mathbf{B})$ *and all* N, $\mathrm{PMC}_{\mathscr{A}}(N)$ *is log-space reducible to* $\mathrm{PCSP}(\mathbf{A}, \mathbf{B})$ *where* $\mathscr{A} = \mathrm{Pol}(\mathbf{A}, \mathbf{B})$.

Example

$\mathrm{PMC}_{\mathscr{K}}(6)$ is log-space reducible to $\mathrm{PCSP}(K_3, K_5)$ ($\mathscr{K} = \mathrm{Pol}(K_3, K_5)$).

The gap

Given that $\mathscr{A} = \mathrm{Pol}(\mathbf{A}, \mathbf{A}')$ satisfies all Mal'cev conditions satisfied in $\mathscr{B} = \mathrm{Pol}(\mathbf{B}, \mathbf{B}')$, we have log-space reductions:

$$\mathrm{PCSP}(\mathbf{B}, \mathbf{B}') \to \mathrm{PMC}_{\mathscr{B}}(N) \to \mathrm{PMC}_{\mathscr{A}}(N) \to \mathrm{PCSP}(\mathbf{A}, \mathbf{A}').$$

Example

$$\mathrm{PCSP}(\mathrm{NAE}_2, \mathrm{NAE}_K) \to \mathrm{PMC}_{\mathscr{H}_K}(6) \to \mathrm{PMC}_{\mathscr{K}}(6) \to \mathrm{PCSP}(K_3, K_5)$$

Fact. Basically, the only 6-ary Mal'cev condition that is not satisfied in $\mathscr{H}_K$ is:

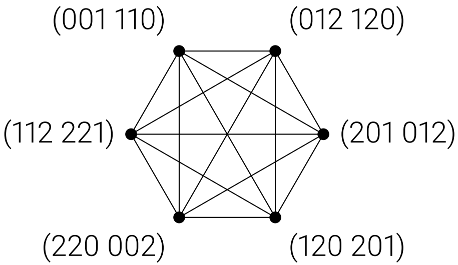
$$o(x, x, y, y, y, x) \approx s(x, y)$$
$$o(x, y, x, y, x, y) \approx s(x, y)$$
$$o(y, x, x, x, y, y) \approx s(x, y).$$

Proof: A graph that is not 5-colorable

$\mathrm{Pol}(K_3, K_5)$ does not have such polymorphism o, such polymorphism is a 5-coloring of

$$K_3^6 \,/\, (x, y, y, y, x, x) \sim (y, x, y, x, y, x) \sim (y, y, x, x, x, y).$$

But that graph contains a 6-clique:



Finale

Theorem
$\mathrm{PCSP}(K_d, K_{2d-1})$ *is NP-hard.*

- In the proof, we did not come with a new source of hardness. We still essentially use the PCP Theorem [Arora, Safra, '98].
- Find a new better proof of the PCP Theorem!

Theorem
If every height 1 Mal'cev condition satisfied by $\mathrm{Pol}(\mathbf{A}, \mathbf{B})$ *is satisfied in* $\mathrm{Pol}(\mathbf{C}, \mathbf{D})$ *then* $\mathrm{PCSP}(\mathbf{C}, \mathbf{D})$ *is log-space reducible to* $\mathrm{PCSP}(\mathbf{A}, \mathbf{B})$.

- Unlike CSP, there is not a single source of hardness of PCSP under algebraic reductions!
- Something is missing.
- Can we use some ideas in approximation, UGC?